Lecture 2
EVOLUTIONARY GAME THEORY
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ties of ESS

0.1 The cardinality of Δ^{ESS}

1 General proper

Proposition 1.1 (Haigh, 1975) *The set Δ^{ESS} is finite.*

Proof sketch:

1. If $x \in \Delta^{ESS}$ then its support contains no other ESS

2. The game has finitely many pure strategies and hence finitely many possible supports

- Recall that the empty set is finite and that some games have no ESS

- Recall that all finite games have Nash equilibria (in pure or mixed strategies) and that this set may be infinite

1.1 Uniform invasion barrier

- Each ESS has a *uniform invasion barrier*:

Proposition 1.2 *$x \in \Delta$ is an ESS $\Rightarrow \exists\, \bar{\varepsilon} \in (0,1)$ such that for all $\varepsilon \in (0,\bar{\varepsilon})$ and all $y \neq x$:*

$$\pi\left[x, \varepsilon y + (1-\varepsilon)x\right] > \pi\left[y, \varepsilon y + (1-\varepsilon)x\right].$$

- Conceptually important because any real population is finite. In a population of size N, the smallest mutant population share is $1/N$

1.2 Local superiority

- Note that an *interior* ESS earns a higher payoff against all mutants than these earn against *themselves:* a form of "global superiority"

Definition 1.1 $x \in \Delta$ *is* **locally superior** *if it has a neighborhood* B *s.t.* $\pi(x, y) > \pi(y, y)\ \forall y \neq x,\ y \in B$.

Proposition 1.3 (Hofbauer, Schuster and Sigmund) $x \in \Delta^{ESS} \Leftrightarrow x$ *is locally superior.*

1.3 Relations to non-cooperative solution concepts

- Evolutionary stability not only implies Nash equilibrium:

Proposition 1.4 $x \in \Delta^{ESS} \Rightarrow x$ *undominated.*

- Hence: $x \in \Delta^{ESS} \Rightarrow (x, x)$ ("trembling hand") perfect equilibrium

- One can also prove the following result:

Proposition 1.5 $x \in \Delta^{ESS} \Rightarrow (x, x)$ *proper equilibrium.*

- *Perfect equilibrium* (Selten, 1975) requires robustness to small probabilities of mistakes.

- *Proper equilibrium* (Myerson, 1978) is a refinement of perfection that requires robustness to small probabilities to mistakes, when less costly mistakes are an order of magnitude more likely than more costly mistakes

- Every finite game has at least one proper equilibrium (and hence also at least one perfect equilibrium)

- van Damme (1984) proved the amazing result that, given any finite normal-form game, and any proper equilibrium in the game: the proper equilibrium induces a *sequential equilibrium* (Kreps and Wilson, 1982) in every extensive-form game with that normal form (see also Kohlberg and Mertens, 1986)

2 Other evolutionary stability concepts

2.1 Neutral stability

- Weak payoff inequality instead of strict:

Definition 2.1 $x \in \Delta$ *is a* **neutrally stable strategy (NSS)** *if for every strategy* y $\exists$ $\bar{\varepsilon}_y \in (0,1)$ *such that for all* $\varepsilon \in (0, \bar{\varepsilon}_y)$*:*

$$\pi\left[x, \varepsilon y + (1-\varepsilon)x\right] \geq \pi\left[y, \varepsilon y + (1-\varepsilon)x\right].$$

- Sometimes neutral stability is called weak evolutionary stability (and sometimes these are mixed up)

- Clearly $\Delta^{ESS} \subset \Delta^{NSS} \subset \Delta^{SNE}$

- There are games with no NSS:

Example 2.1

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

In this game a strategy that is a best reply to itself is a worse reply to its alternative best replies than they are to themselves.

2.2 Robustness against equilibrium entrants

- Requiring robustness only against "rational" mutants, mutants that are optimal in the post-entry population

Definition 2.2 (Swinkels, 1992) $x \in \Delta$ *is* **robust against equilibrium entrants (REE)** *if* $\exists\ \bar{\varepsilon} \in (0,1)$ *such that for all* $\varepsilon \in (0,\bar{\varepsilon})$ *and* $y \neq x$*:*

$$y \notin \beta^* [\varepsilon y + (1-\varepsilon)x]$$

- There are games with no REE (for example, when all payoffs are the same)

- Since ESS have uniform invasion barriers:

$$\Delta^{ESS} \subset \Delta^{REE} \subset \Delta^{SNE}$$

2.3 Evolutionarily stable sets of strategies

- Thomas (1985)

Definition 2.3 *A non-empty and closed set X is an* **evolutionarily stable set** *(an* **ES set***) if there each $x \in X$ has some neighborhood B such $\pi(x, y) \geq \pi(y, y)$ for all $y \in B$, with strict inequality if $y \notin X$.*

- $\{x^*\}$ is an ES set iff $x^* \in \Delta^{ESS}$ (since ESS is equivalent with local superiority)

- X evolutionarily stable $\Rightarrow X \subset \Delta^{NSS}$

Proposition 2.1 *(i) $X \subset \Delta^{ESS} \Rightarrow X$ is an ES set, (ii) X, X' ES sets $\Rightarrow X \cup X'$ is an ES set, (iii) $X \cup X'$ is an ES set and $X \cap X' \neq \varnothing$, then X and X' are ES sets.*

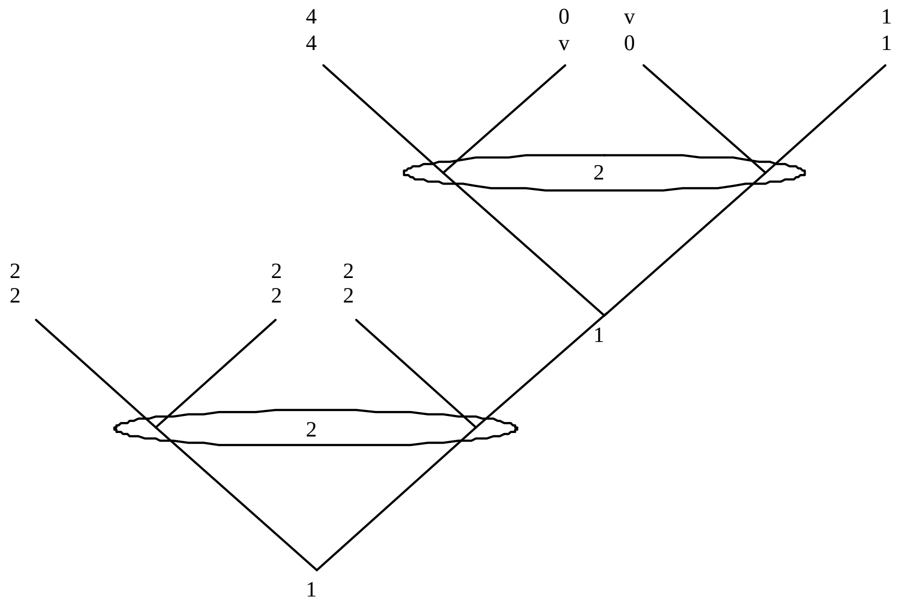
2.4 Equilibrium-evolutionary stable sets

- Setwise robustness against "equilibrium entrants:"

Definition 2.4 (Swinkels, 1993) *A set* $X \subset \Delta$ *is an* **equilibrium evolutionarily stable (EES)** *if it is minimal with respect to the following property:*

X *is a non-empty and closed subset of* Δ^{SNE} *for which* $\exists\ \bar{\varepsilon} \in (0,1)$ *such that if* $x \in X$, $y \in \Delta$, $\varepsilon \in (0, \bar{\varepsilon})$ *and* $y \in \tilde{\beta}\left[\varepsilon y + (1-\varepsilon)x\right]$, *then* $\varepsilon y + (1-\varepsilon)x \in X$.

Example 2.2 *Entry with veto right to a prisoner's dilemma (let $v > 4$):*



Pure-strategy normal form payoff matrix (A=abstain, E=enter, C=cooperate, D=defect):

	AC	AD	EC	ED
AC	2	2	2	2
AD	2	2	2	2
EC	2	2	4	0
ED	2	2	v	1

- To play A, that is to say "no thank you" to the suggestion to play the PD, seems reasonable, but:

1. Is A compatible with ESS? NSS? [no, because mutants who say "yes please" and play C in the PD can invade]

2. Is A compatible with EES? [yes, because the above mutants are not behaving optimally in the post-entry population]

3 The replicator dynamic

[Taylor and Jonker, 1978]

- Domain of analysis the same as for ESS: finite and symmetric two-player games

Heuristically:

1. A population of individuals who are recurrently and randomly matched in pairs to play the game

2. Individuals use only *pure strategies* (like in Nash's mass-action interpretation)

3. A mixed strategy is now interpreted as a *population state*, a vector of populations shares

4. Population shares change, depending on the *current average payoff* to each pure strategy

5. The changes are described by a *system of ordinary differential equations*

Formally:

- Again a large (continuum) population playing a symmetric finite game
- But now each individual always plays a pure strategy
- At each time $t \in \mathbb{R}$, and for each $h \in S$, let $x_h(t)$ be the population share of h-*strategists* (individuals who use pure strategy h)
- *Population state*: $x(t) = (x_1(t), ..., x_m(t)) \in \Delta$

- Expected payoff to pure strategy h at a random match (with $e^h \in \boldsymbol{\Delta}$ denoting the h^{th} unit vector):

$$\pi(e^h, x) = e^h \cdot Ax$$

- *Population average payoff* :

$$\pi(x, x) = \sum_{h \in S} x_h \pi(e^h, x)$$

The replicator dynamic:

$$\dot{x}_h = \left[\pi(e^h, x) - \pi(x, x)\right] \cdot x_h \quad \forall h \in S$$

- *Growth rate* of population shares:

$$\dot{x}_h / x_h = \pi(e^h, x) - \pi(x, x)$$

- Better (worse) than-average strategies grow (decline) and *best* replies have the highest growth rate

3.1 Solving the replicator dynamic

- Polynomial *vector field*

$$f_h(x) = \left[\pi(e^h, x) - \pi(x, x)\right] x_h$$

- Picard-Lindelöf Theorem: $\exists!$ (global) *solution* $\xi : \mathbb{R} \times \mathbf{\Delta} \rightarrow \mathbf{\Delta}$ through any initial state $x^o \in \mathbf{\Delta}$

- Here $x = \xi(t, x^o)$ is the population state at time t if the initial state was x^o

Dynamic stability concepts

- A population state x is *Lyapunov stable* if small perturbations does not initiate a movement away from x. [Formally: for every neighborhood B of x there should exist a subneighborhood $B^o \subset B$ of x such that if $x^o \in B^o$ then $\xi(t, x^o) \in B$ for all $t > 0$.]

- A population state is *asymptotically stable* if it is Lyapunov stable and, moreover, the population returns asymptotically (over time) towards x after any sufficiently small perturbation. [Formally: in addition to Lyapunov stability, x should have a neighborhood A such that $x^o \in A \Rightarrow \xi(t, x^o) \to x$ as $t \to +\infty$.]

3.2 Connection to ESS

Proposition 3.1 *If $x \in \Delta^{ESS}$, then x is asymptotically stable in the replicator dynamic*

- The converse holds for 2×2 games, but not in general

- Counter-example in class

3.3 Connections to non-cooperative game theory

Proposition 3.2 *(a)* $x \in \Delta$ *Lyapunov stable* $\Rightarrow x \in \Delta^{SNE}$, *(b)* $x^o \in int(\Delta) \wedge \lim_{t\to+\infty} \xi(t, x^o) = x \Rightarrow x \in \Delta^{SNE}$, *(c)* $h \in S$ *strictly dominated* $\Rightarrow \lim_{t\to+\infty} \xi_h(t, x^o) = 0 \ \forall x^o \in int(\Delta)$.

- Note that the third result
  - does not presume that the solution trajectory converges
  - can be strengthened to $h \in S$ not rationalizable
- Hence, it is as if, asymptotically over time, CK[game+rationality] would hold!

Proof sketch for (c): Suppose $k \in S$ is strictly dominated by $y \in \Delta$

1. By continuity

$$\min_{x \in \Delta} \left[\pi(y, x) - \pi(e^k, x)\right] = \delta > 0$$

2. Let $V : int(\Delta) \to \mathbb{R}$ be defined by

$$V(x) = \sum_{h \in S} y_h \ln(x_h) - \ln(x_k)$$

3. Then

$$\dot{V}(x) = \sum_{h \in S} \frac{\partial V(x)}{\partial x_h} \dot{x}_h = \sum_{h \in S} \frac{y_h \dot{x}_h}{x_h} - \frac{\dot{x}_k}{x_k} \geq \delta \quad \forall x \in \Delta$$

4. Hence, V increases towards $+\infty$ along the solution trajectories, so $\xi_k(t, x^o) \to 0$ for all $x^o \in int(\Delta)$.

Other results for the replicator dynamic:

Proposition 3.3 *(a)* $x \in \Delta$ *asymptotically stable* $\Rightarrow (x, x)$ *is a ("trembling-hand") perfect equilibrium, (b)* $x \in \Delta^{NSS} \Rightarrow x$ *Lyapunov stable, (c)* X *an ES set* $\Rightarrow X$ *asymptotically stable (as a set).*

THE END

Literature: Chapter 9 in van Damme (1991) or Chapters 2, 3 in Weibull (1995).