Lecture 3
EVOLUTIONARY GAME THEORY
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November 15, 2012

1 Multi-population selection dynamics

This setting is close to that in Nash's (1950) mass-action interpretation

- Domain: finite normal-form games $G = \langle I, S, u \rangle$
  - $I = \{1, .., n\}$ the set of *player roles*
  - $S = \times_{i \in I} S_i$ the set of *pure-strategy profiles,* $s = (s_1, s_2, .., s_n)$
  - $u : S \to \mathbb{R}^n$ the *combined payoff function*, $u_i(s)$ being the payoff to the individual in player role i

- A continuum population for each player role i

- All individuals play pure strategies
- Let $\Delta(S_i)$ denote the unit simplex of mixed strategies for player i
- Let $\boxdot(S)$ be the polyhedron of mixed-strategy profiles, $\boxdot(S) = \times_{i \in I} \Delta(S_i)$
- Extend u from S to $\boxdot(S)$ in the usual way and write $\tilde{u} : \boxdot(S) \rightarrow \mathbb{R}^n$
- Thus $\tilde{u}_i(x)$ is player i's (expected) payoff when mixed-strategy profile $x = (x_1, .., x_n)$ is played

1.1 The replicator dynamic

Taylor (1979), a follow-up of Taylor and Jonker (1978)

- For each player role $i \in I$ and pure strategy $h \in S_i$ (with $e_i^h \in \Delta(S_i)$ placing unit probability on h):

$$\dot{x}_{ih} = \left[\tilde{u}_i(e_i^h, x_{-i}) - \tilde{u}_i(x)\right] x_{ih} \ .$$

- The *growth rate* of the subpopulation of h-strategists in player population i:

$$g_{ih}(x) = \tilde{u}_i(e_i^h, x_{-i}) - \tilde{u}_i(x)$$

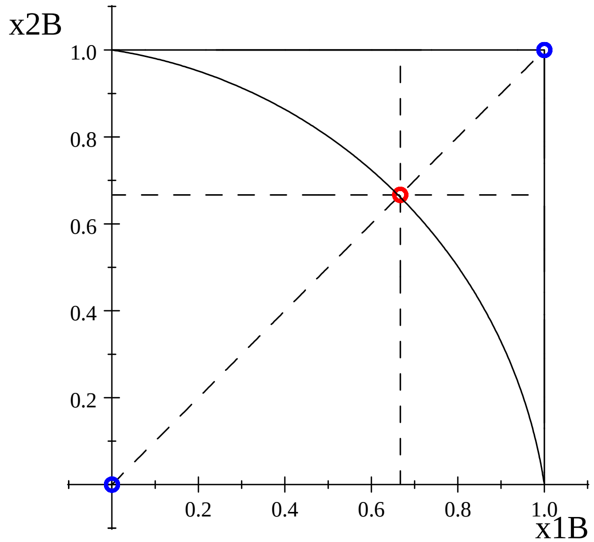
1.2 Two-player games

Bi-matrix (A, B):

$$\begin{cases} \dot{x}_{1h} = \left[e_1^h \cdot Ax_2 - x_1 \cdot Ax_2\right] x_{1h} & \forall h \in S_1 \\ \dot{x}_{2k} = \left[e_2^k \cdot B^T x_1 - x_2 \cdot B^T x_1\right] x_{2k} & \forall k \in S_2 \end{cases}$$

Example 1.1 *Matching-pennies game has periodic solutions, orbits around the NE the state space. The NE is hence Lyapunov stable but not asymptotically stable.*

Example 1.2 *Coordination game. Each strict equilibrium is asymptotically stable and the mixed equilibrium is unstable.*



Example 1.3 (entry deterrence) *Player 1 is a potential entrant into a market where player 1 initially is a monopolist. The potential entrant may enter, E, or abstain, A. Upon entry, the monopolist may fight, F, for example by flooding the market, or yield, C, for example by splitting the market.*

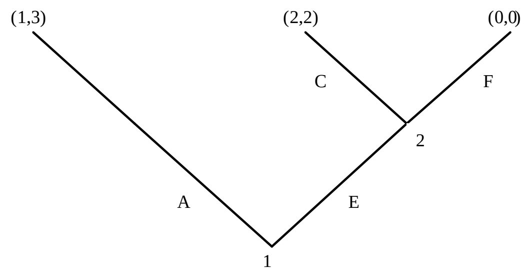


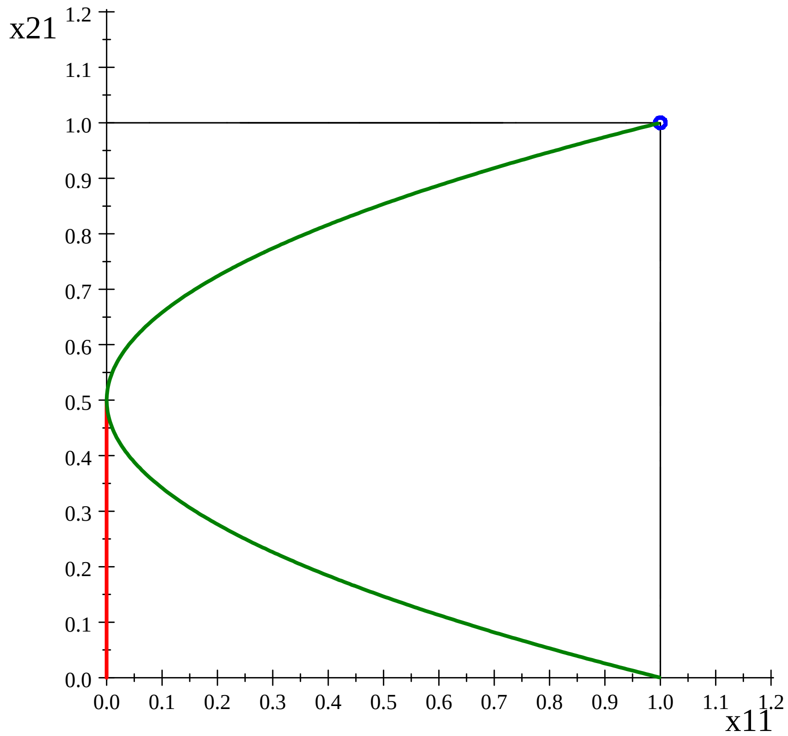
Figure 1:

There is a unique subgame-perfect equilibrium, (E,C), but infinitely many other Nash equilibria.

Its normal form:

	C	F
A	$1,3$	$1,3$
E	$2,2$	$0,0$

The two-population (Taylor) replicator dynamic:



1.3 General selection dynamics

Consider population dynamics of the form

$$\dot{x}_{ih} = g_{ih}(x)\, x_{ih}$$

where g is *regular* ($=$ Lipschitz continuous and $x \cdot g(x) \equiv 0$).

Definition 1.1 *A regular growth-rate function g is* **payoff-positive (PP)** *if, for all $x \in \boxdot$ and $i \in I$:*

$$g_{ih}(x) \begin{array}{c} < \\ = \\ > \end{array} 0 \;\Leftrightarrow\; \tilde{u}_i(e_i^h, x_{-i}) \begin{array}{c} < \\ = \\ > \end{array} \tilde{u}_i(x)$$

- Let

$$B_i\left(x\right) = \left\{h \in S_i : \tilde{u}_i(e_i^h, x_{-i}) > \tilde{u}_i(x)\right\}$$

  - the pure strategies that yield payoffs above average. (The empty set in Nash equilibrium.)

Definition 1.2 *A regular growth-rate function* g *is* **weakly payoff-positive (WPP)** *if, for all* $x \in \boxdot$ *and* $i \in I$*:*

$$B_i(x) \neq \varnothing \ \Rightarrow g_{ih}(x) > 0 \text{ for some } h \in B_i(x).$$

Proposition 1.1 *For any weakly payoff-positive dynamic:*

(a) $x \in \boxdot$ *Lyapunov stable* $\Rightarrow x \in \boxdot^{NE}$

(b) $x^o \in int\,(\boxdot) \wedge \lim_{t \to +\infty} \xi\,(x^o, t) = x^* \quad \Rightarrow \quad x^* \in \boxdot^{NE}$

Definition 1.3 *A regular growth-rate function* g *is* **convex monotonic** *if each* $g_{ih}(x)$ *is increasing and convex in* $\tilde{u}_i(e_i^h, x_{-i})$.

- Example: the Taylor multi-population replicator dynamic.

Proposition 1.2 (Hofbauer and Weibull, 1996) *For any convex-monotonic dynamic:*

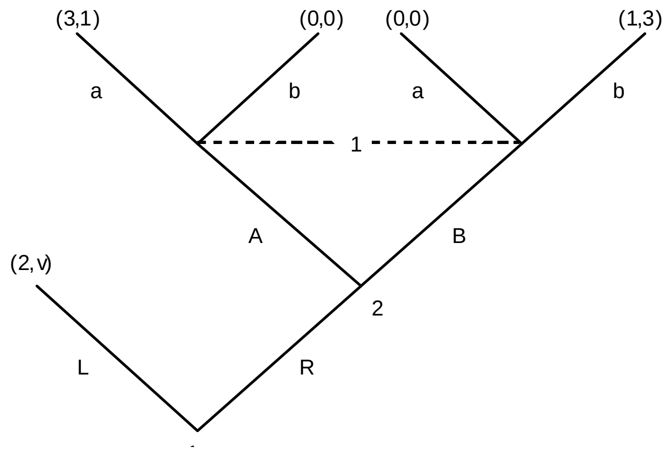
(c) $h \in S$ *iteratively strictly dominated* $\Rightarrow \lim_{t\to+\infty} \xi_h(x^o, t) = 0 \; \forall x^o \in int(\boxdot)$

- Convex monotonicity in fact not only sufficient, but actually almost necessary for (c)

- If $n = 2$, then (c) $\Rightarrow$ all non-rationalizable pure strategies vanish asymptotically over time

  As if CK[game+rationality] would hold!

Example 1.4 (outside-option game) *Players 1 has an outside option (go to a café with a third person) or interact with player 2. In the second case, players 1 and 2 may either go to the opera (options A and a), something player 1 would prefer, or else go to a rugby match (options B and b), something player 2 would prefer. Player 2 will learn whether or not player 1 has taken her outside option, but they cannot communicate (cell phone battery is out). What is your prediction?*



Multiple sequential equilibria. Only one is compatible with "forward-induction".

What happens in PP selection dynamics? Consider the reduced NF:

	A	B
L	$2, v$	$2, v$
Ra	$3, 1$	$0, 0$
Rb	$0, 0$	$1, 3$

Unique asymptotically stable state: $x^* = (Ra, A)$

2 Stochastic population processes

[Benaïm and Weibull (2003)]

- Domain: all finite normal-form games

- We have studied *deterministic* population dynamics in *continuous* time with *continuum* populations

- We now study *stochastic* population processes in *discrete* time with *finite* populations

2.1 The population process

- One player population, of size N, for each player role

- All individuals play pure strategies

- Random draw of 1 individual for *strategy review*, at discrete times $t = 0$, $1/N$, $2/N$...

- Equal probability for each individual to be drawn

- *Population state*: vector $X(t) = \langle X_1(t), ..., X_n(t) \rangle$ of player-population vectors $X_i(t) = (X_{ih}(t))_{h \in S_i}$ where $X_{ih}(t) = N_{ih}(t)/N$

Define a *Markov chain* $X^N = \left\langle X^N(t) \right\rangle$ on

$$\Theta^N = \{x \in \boxdot(S) : Nx_{ih} \in \mathbb{N} \quad \forall i \in I, h \in S_i\}$$

as follows:

1. For all player roles $i \in I$ and pure strategies $h, k \in S_i$, let $p_{ik}^{hN} : \boxdot(S) \to [0,1]$ be a Lipschitz continuous *transition probability* function (from pure strategy k to pure strategy h):

$$\Pr\left[X_i^N(t + \frac{1}{N}) = x_i + \frac{1}{N}\left(e_i^h - e_i^k\right) \mid X^N(t) = x\right] = p_{ik}^{hN}(x)$$

   with $x_{ik} = 0 \Rightarrow p_{ik}^{hN}(x) = 0$

2. The *expected net increase* in subpopulation (i,h), from t to $t + 1/N$, conditional upon the current state x:

$$F_{ih}^N(x) = \sum_{k \neq h} p_{ik}^{hN}(x) - \sum_{k \neq h} p_{ih}^{kN}(x) \ .$$

3. Assume that

   (a) F^N bounded

   (b) $\exists$ common Lipschitz constant $\forall F^N$

   (c) $F^N \to F$ uniformly

4. Then also F is bounded and Lipschitz continuous

- We are interested in deterministic continuous-time, continuum population approximation of X^N when N is large

2.2 Mean-field equations

The system of *mean-field* equations:

$$\dot{x}_{ih} = F_{ih}(x) \qquad \forall i, h, x$$

- *Solution mapping* $\xi : \mathbb{R} \times \boxdot(S) \to \boxdot(S)$

- Affine interpolation of the process X^N: $\hat{X}^N$ (connect the points by straight-line segments)

- Deviation between the flow ξ and $\hat{X}^N$ at any time $t \in \mathbb{R}$:

$$||\hat{X}^N(t) - \xi(t, x)|| = \max_{i \in I, h \in S_i} \left| \hat{X}^N_{ih}(t) - \xi_{ih}(t, x) \right|$$

- The *maximal deviation* on bounded time interval $[0, T]$:

$$D^N(T, x) = \max_{0 \le t \le T} ||\hat{X}^N(t) - \xi(t, x)||$$

Proposition 2.1 $\forall T > 0 \, \exists \, c > 0$ *such that* $\forall \varepsilon > 0$ *and any* N *large enough:*

$$\Pr\left[D^N(T, x) \ge \varepsilon \mid X^N(0) = x\right] \le 2Me^{-\varepsilon^2 cN} \qquad \forall x \in \boxdot^N(S).$$

- Here $M = m - n$ (the dimension of the so-called tangent space of $\boxdot(S)$)

- This result can be used to establish result that connect the behavior of the stochastic population process X^N, for N large, with properties of its deterministic mean-field

- If the mean-field takes the form of a selection dynamic of the types studied above, then we can probabilistically predict the stochastic population process!

2.3 Exit times

Definition 2.1 *First exit time from a set $B \subset \boxdot(S)$:*

$$\tau^N(B) = \inf\left\{t \geq 0 : \hat{X}^N(t) \notin B\right\} \ .$$

- Consider the *forward orbit* of the mean-field solution ξ through an initial state x^0:

$$\gamma^+(x^0) = \{x \in \boxdot(S) : x = \xi(t, x^o) \text{ for some } t \geq 0\}$$

Proposition 2.2 *Let B be an open neighborhood of the closure of $\gamma^+(x^0)$ and suppose that $X^N(0) \to x^0$. Then*

$$\Pr\left[\lim_{N\to\infty} \tau^N(B) = +\infty\right] = 1$$

- In particular, if (a) $x^0 \in int\,[\boxdot(S)]$, (b) the mean-field is WPP, and (c) $\xi\left(t, x^0\right) \to x^*$, then we know from the above that $x^* \in \boxdot^{NE}(S)$. Hence, this proposition then says that $X^N(t)$, for large enough N, will stay close to the trajectory of ξ, move towards the NE x^* and remain close to it for a very long time

Definition 2.2 *The* **basin of attraction** *of a closed asymptotically stable set $A \subset \boxdot(S)$ in the mean field ξ: the set*

$$\mathcal{B}(A) = \{x^o \in \boxdot(S) : \xi(t, x^o)_{t\to\infty} \to A\}$$

Proposition 2.3 *Let $A \subset \boxdot(S)$ be closed and asymptotically stable set in the mean field ξ. Every neighborhood $B_1 \subset \mathcal{B}(A)$ of A contains a sub-neighborhood B_0 of A s.t.*

$$X^N(0) \in B_0 \; \forall N \quad \Rightarrow \quad \Pr\left[\liminf_{N\to\infty} \tau^N(B_1) = +\infty\right] = 1$$

- Hence, strategy profiles and closed sets of strategy profiles that are asymptotically stable in the deterministic mean-field are "stochastically robust" in the population process when the population is large.

2.4 Visitation rates

- The empirical *visitation rate* to any given subset of strategy profiles is the time share spent in the set

- One can prove that almost all of the time the stochastic process will "hang around" the so-called *Birkhoff center* of its mean-field

- If the stochastic population process is *ergodic*, then one can make more precise long-run predictions, even selection among strict Nash equilibria

THE END

Literature: Chapters 3 and 5 in Weibull (1995), and Benaim and Weibull (2003).